

MULTIVARIATE IGUSA THEORY: DECAY RATES OF EXPONENTIAL SUMS

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ABSTRACT. We obtain general estimates for exponential integrals of the form

$$E_f(y) = \int_{\mathbb{Z}_p^n} \psi\left(\sum_{j=1}^r y_j f_j(x)\right) |dx|,$$

where the f_j are restricted power series over \mathbb{Q}_p , $y_j \in \mathbb{Q}_p$, and ψ a nontrivial additive character on \mathbb{Q}_p . We prove that if (f_1, \dots, f_r) is a dominant map, then $|E_f(y)| < c|y|^\alpha$ for some $c > 0$ and $\alpha < 0$, uniform in y , where $|y| = \max(|y_i|)_i$. In fact, we obtain similar estimates for a much bigger class of exponential integrals. To prove these estimates we introduce a new method to study exponential sums, namely, we use the theory of p -adic subanalytic sets and p -adic integration techniques based on p -adic cell decomposition. We compare our results to some elementarily obtained explicit bounds for E_f with f_j polynomials.

1. INTRODUCTION

For $f = (f_1, \dots, f_r)$ an r -tuple of restricted power series over \mathbb{Q}_p in the variables $x = (x_1, \dots, x_n)$ and for $y \in \mathbb{Q}_p^r$, we consider the exponential integral

$$E_f(y) = \int_{\mathbb{Z}_p^n} \psi(y \cdot f(x)) |dx|,$$

where ψ is a nontrivial additive character on \mathbb{Q}_p , $|dx|$ denotes the normalized Haar measure on \mathbb{Q}_p^n , and $y \cdot f(x) = \sum_j y_j f_j(x)$.

With $|y| = \max(|y_i|)_i$ and \ll the Vinogradov symbol, we obtain the following general upper bounds:

Theorem 1.1. *If $f(\mathbb{Z}_p^n)$ has nonempty interior in \mathbb{Q}_p^r , there exists a real number $\alpha < 0$ such that*

$$E_f(y) \ll \min\{|y|^\alpha, 1\}.$$

In his book [10] of 1978, J. Igusa proves Theorem 1.1 in the case that $r = 1$ with $f = f_1$ a nonconstant homogeneous polynomial, and he formulates the problem of generalizing this to the case of $r > 1$. In this case Igusa is also able to give an explicit $\alpha < 0$ in terms of the numerical data of an embedded resolution of f . By a very fine analysis of embedded resolutions of f , Lichtin [11] is able to prove

2000 *Mathematics Subject Classification.* Primary 11L07, 11U09, 32B20; Secondary 11L05, 11S80, 32P05, 32B20, 03C10.

Key words and phrases. exponential sums, Kloosterman sums, subanalytic p -adic geometry, p -adic cell decomposition, p -adic integrals, local singular series, Igusa's local zeta function, circle method.

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Theorem 1.1 in the case of a dominant map of $r = 2$ polynomials, where he also gives an explicit $\alpha < 0$ in terms of the geometry of f . At present, these proofs seem to be difficult to be generalized to the case of $r > 2$ polynomials. In the case of one nonconstant polynomial $f = f_1$, Theorem 1.1 can be proven by elementary methods, see for instance the work of Chubarikov [1] and Loxton [13]. In the last section we show how the results of [1] can be used to derive Theorem 1.1, when f_1, \dots, f_r are polynomials, in a very short way (even with explicit upper bounds and weaker suppositions). We indicate there why the situation for analytic maps is more difficult.

1.1. In this paper we present a new technique to study exponential integrals of a general nature, namely, by studying rather general p -adic integrals by means of p -adic cell decomposition and the theory of subanalytic sets. Examples of such general exponential integrals are given below in this introduction. These techniques are also used in other contexts, for example, by Denef [3] to prove the rationality of the Serre-Poincaré series associated to the p -adic points on a variety.

1.2. For readers not familiar with p -adic integration, we indicate how $E_f(y)$ can be understood as an exponential sum. In the case that the f_i are restricted power series over \mathbb{Z}_p , $\psi(x) = \exp(2\pi i(x \bmod \mathbb{Z}_p))$ (abbreviated by $\exp(2\pi i x)$), and

$$(1) \quad y = \left(\frac{u_1}{p^m}, \dots, \frac{u_r}{p^m} \right),$$

with u_i integers satisfying $(u_1, \dots, u_r, p) = 1$, $m \geq 0$, we can write

$$E_f(y) = \frac{1}{p^{mn}} \sum_{x \in (\mathbb{Z}_p/p^m)^n} \exp(2\pi i \frac{\sum_{j=1}^r u_j f_j(x)}{p^m}).$$

Note that for general $y' \in \mathbb{Q}_p^r$ there can always be found a tuple y of the form (1) such that $E_f(y') = E_f(y)$. Theorem 1.1 then says that $|E_f(y)|$ can be bounded by $cp^{m\alpha}$ for some $c > 0$ and $\alpha < 0$, uniform in y .

1.3. We use the notion of subanalytic sets as in [8] and the recent notion of subanalytic constructible functions as in [2] (see below for the definitions).

Let $G : \mathbb{Q}_p^r \rightarrow \mathbb{Q}$ be an integrable subanalytic constructible function, and let $G^*(y) := \int_{\mathbb{Q}_p^n} G(x) \psi(x \cdot y) |dx|$ be its Fourier transform. We obtain the following general upper bounds:

Theorem 1.2. *There exists a real number $\alpha < 0$ such that $G^*(y) \ll \min\{|y|^\alpha, 1\}$.*

1.4. We indicate how Theorem 1.1 follows from Theorem 1.2. It is well known that, whenever $f(\mathbb{Z}_p^n)$ has nonempty interior in \mathbb{Z}_p^r , E_f is the Fourier transform of an integrable function $F_f : \mathbb{Q}_p^r \rightarrow \mathbb{Q}$ (see [10] or [15]). We prove that we can take F_f to be a subanalytic constructible function (see Theorem 3.1 below). Theorem 1.1 then follows immediately from Theorem 1.2.

In fact, a similar reasoning leads to the following much more general result:

Theorem 1.3. *If $f : \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p^r$ is a subanalytic map and $\phi : \mathbb{Q}_p^n \rightarrow \mathbb{Q}$ is an integrable subanalytic constructible function such that the support of ϕ is contained*

in $f^{-1}(\text{Regular values of } f) \cup A$, with A a set of measure zero, then there exists a real number $\alpha < 0$ such that $E_{\phi,f}(y) \ll \min\{|y|^\alpha, 1\}$, with

$$E_{\phi,f}(y) := \int_{\mathbb{Q}_p^n} \phi(x) \psi(y \cdot f(x)) |dx|.$$

We end section 5 with an open question about what happens if f is analytic but no longer subanalytic. All results of the paper also hold for finite field extensions of \mathbb{Q}_p .

Possible applications, or, possible subjects for future research, lie in the search for candidate exponents α of Theorem 1.1 using the numerical data of a (parameterized) resolution of singularities of the family $\sum_{i=1}^r u_i f_i$ with parameters u_i (if such resolution exists); as noted before, candidate exponents can be found in this way when $r = 1$, see [6]. Also, one can try to establish, under similar conditions as in [10], an analytic analogue of the Poisson summation formula considered by Igusa [10].

1.5. Notation and terminology. We fix a p -adic field K (i.e. $[K : \mathbb{Q}_p]$ is finite) and write R for the valuation ring of K , π_0 for a uniformizer of R , and q for the cardinality of the residue field. For $x \in K$, $v(x) \in \mathbb{Z} \cup \{\infty\}$ denotes the p -adic valuation of x and $|x| = q^{-v(x)}$ the p -adic norm. We write P_n for the collection of n -th powers in $K^\times = K \setminus \{0\}$, $n > 0$, and $\lambda P_n = \{\lambda x \mid x \in P_n\}$ for $\lambda \in K$. Let ψ be a nontrivial additive character on K . We write $x \cdot y = x_1 y_1 + \dots + x_n y_n$ for $x, y \in K^n$, $n > 0$.

The Vinogradov symbol \ll has its usual meaning, namely that for complex valued functions f and g with g taking non-negative real values $f \ll g$ means $|f| \leq cg$ for some constant c .

A restricted analytic function $R^n \rightarrow K$ is an analytic function, given by a single restricted power series over K in n variables (by definition, this is a power series over K which converges on R^n). We extend each restricted analytic function $R^n \rightarrow K$ to a function $K^n \rightarrow K$ by putting it zero outside R^n . A key notion is the following:

Definition 1.4. A subset of K^n is called (globally) *subanalytic* if it can be obtained in finitely many steps by taking finite unions, intersections, complements and linear projections of zero loci of polynomials and of zero loci of restricted analytic functions in K^{n+e} , $e \geq 0$. A function $f : X \subset K^m \rightarrow K^n$ is called subanalytic if its graph is a subanalytic set.

We recall a basic result on subanalytic sets:

Proposition 1.5 ([8], Proposition (3.29)). *Let $X \subset K^n$ be a subanalytic set and $f : X \rightarrow K$ a subanalytic function. Then there exists a finite partition of X into p -adic submanifolds A_j of K^n such that the restriction of f to each A_j is analytic and such that each A_j is subanalytic.*

We refer to [2], [5, 7], [8], and [9] for the theory of subanalytic sets.

2. CELL DECOMPOSITION AND p -ADIC INTEGRATION

Cell decomposition is well suited to describe piecewise several kinds of p -adic maps, for example, polynomials maps, restricted analytic maps, subanalytic constructible functions, and so on. It allows one to partition the domain of such functions into p -adic manifolds of a simple form, called cells, and to obtain on each

of these cells a nice description of the way the function depends on a specific special variable (for an example of such an application, see Lemma 2.5). By induction one gets a nice description of the function with respect to the other variables.

Cells are defined by induction on the number of variables:

Definition 2.1. A cell $A \subset K$ is a (nonempty) set of the form

$$(2) \quad \{t \in K \mid |\alpha| \square_1 |t - c| \square_2 |\beta|, \ t - c \in \lambda P_n\},$$

with constants $n > 0$, $\lambda, c \in K$, $\alpha, \beta \in K^\times$, and \square_i either $<$ or no condition. A cell $A \subset K^{m+1}$, $m \geq 0$, is a set of the form

$$(3) \quad \{(x, t) \in K^{m+1} \mid x \in D, \ |\alpha(x)| \square_1 |t - c(x)| \square_2 |\beta(x)|, \\ t - c(x) \in \lambda P_n\},$$

with $(x, t) = (x_1, \dots, x_m, t)$, $n > 0$, $\lambda \in K$, $D = \pi_m(A)$ a cell where π_m is the projection $K^{m+1} \rightarrow K^m$, subanalytic functions $\alpha, \beta : K^m \rightarrow K^\times$ and $c : K^m \rightarrow K$, and \square_i either $<$ or no condition, such that the functions α, β , and c are analytic on D . We call c the center of the cell A and λP_n the coset of A .

Note that a cell is either the graph of an analytic function defined on D (namely if $\lambda = 0$), or, for each $x \in D$, the fiber $A_x = \{t \mid (x, t) \in A\}$ is a nonempty open (if $\lambda \neq 0$).

Theorem 2.2 (*p*-adic cell decomposition, [2]). *Let $X \subset K^{m+1}$ be a subanalytic set and $f_j : X \rightarrow K$ subanalytic functions for $j = 1, \dots, r$. Then there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that*

$$(4) \quad |f_j(x, t)| = |\delta_{ij}(x)| \cdot |(t - c_i(x))^{a_{ij}} \lambda_i^{-a_{ij}}|^{\frac{1}{n_i}}, \quad \text{for each } (x, t) \in A_i,$$

with $(x, t) = (x_1, \dots, x_m, t)$, integers a_{ij} , and $\delta_{ij} : K^m \rightarrow K$ subanalytic functions, analytic on $\pi_m(A_i)$, $j = 1, \dots, r$. If $\lambda_i = 0$, we use the convention that $a_{ij} = 0$.

Theorem 2.2 is a generalisation of cell decomposition for polynomial maps by Denef [3], [4]. Recently, in [7] and [2], cell decomposition has been used to study parametrized integrals, as follows.

Definition 2.3. For each subanalytic set X , we let $\mathcal{C}(X)$ be the \mathbb{Q} -algebra generated by the functions $|h|$ and $v(h)$ for all subanalytic functions $h : X \rightarrow K^\times$. We call $G \in \mathcal{C}(X)$ a *subanalytic constructible function* on X .

To any function G in $\mathcal{C}(K^{m+n})$, $m, n \geq 0$, we associate a function $I_m(G) : K^m \rightarrow \mathbb{Q}$ by putting

$$(5) \quad I_m(G)(x) = \int_{K^n} G(x, y) |dy|$$

if the function $y \mapsto G(x, y)$ is absolutely integrable for all $x \in K^m$, and by putting $I_m(G)(x) = 0$ otherwise.

Theorem 2.4 (Basic Theorem on *p*-adic Analytic Integrals [2]). *For any function $G \in \mathcal{C}(K^{m+n})$, the function $I_m(G)$ is in $\mathcal{C}(K^m)$.*

Lemma 2.5. *Let $X \subset K^{m+1}$ be a subanalytic set and let G_j be functions in $\mathcal{C}(X)$ in the variables (x_1, \dots, x_m, t) for $j = 1, \dots, r$. Then there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that each restriction $G_j|_{A_i}$ is a finite sum of functions of the form*

$$|(t - c_i(x))^a \lambda^{-a}|^{\frac{1}{n_i}} v(t - c_i(x))^s h(x),$$

where $h : K^m \rightarrow \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers. Also, for any function $G \in \mathcal{C}(K^n)$ there exists a closed subanalytic set $A \subset K^n$ of measure zero such that G is locally constant on $K^n \setminus A$.

Proof. The description is immediate from Theorem 2.2 and the definitions. The statement about $G \in \mathcal{C}(K^n)$ follows from Proposition 1.5 and the definitions. \square

The following corollary is immediate.

Corollary 2.6. *Let G be in $\mathcal{C}(K)$. Suppose that if $|y|$ tends to ∞ then $G(y)$ converges to zero. Then there exists a real number $\alpha < 0$ such that $G(y) \ll |y|^\alpha$.*

We prove the following addendum to Theorem 2.4:

Proposition 2.7. *Let G in $\mathcal{C}(K^{r+n})$ be such that $G(x, \cdot) : K^n \rightarrow \mathbb{Q}$ is integrable for almost all $x \in K^r$. Then, there exists a function $F \in \mathcal{C}(K^r)$ such that for all $x \in K^r \setminus B$, with B a subanalytic set of measure zero, one has*

$$F(x) = \int_{K^n} G(x, y) |dy|.$$

Proof. By induction and by Fubini's theorem it is enough to treat the case $n = 1$.

By Lemma 2.5, we can partition K^{r+1} into cells A with center c and coset λP_m such that $G|_A$ is a finite sum of functions of the form

$$(6) \quad H(x, y) = |(y - c(x))^a \lambda^{-a}|^{\frac{1}{m}} v(y - c(x))^s h(x),$$

where $h : K^r \rightarrow \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers.

Claim 1. *Possibly after refining the partition, we can assure that for each A either the projection $A' := \pi_r(A) \subset K^r$ has zero measure, or we can write $G|_A$ as a sum of terms H of the form (6) such that the function $H(x, \cdot)$ is integrable over $A_x := \{y \mid (x, y) \in A\}$ for all $x \in A'$.*

First we prove the claim. By partitioning further, we may suppose that either $v(y - c)$ is constant on A , or, it takes infinitely many values on A , and in the case that $v(y - c)$ is constant on A , we may assume that $a = s = 0$. Regroup the terms with the same exponents (a, s) , by summing up the respective functions h .

By the description (6) of H and by the definition of cells, the fact that the function

$$H(x, \cdot) : A_x \rightarrow \mathbb{Q} : y \mapsto H(x, y)$$

is integrable over A_x only depends on the exponents (a, s) , on the fact whether $h(x)$ is zero or not, and on the particular form of the cell A_x . Also, if terms H_1, \dots, H_k have different exponents (a_i, s_i) , then they have a different asymptotical behavior for y going to $c(x)$ with x fixed, and hence, if their sum is integrable over A_x , then each H_i is integrable over A_x .

Suppose now that A has nonempty interior. Let H be a term with exponents (a, s) and function h as in (6). Then, either $h(x)$ is almost everywhere zero, or, there exists by Lemma 2.5 a nonempty open $U \subset A'$ such that $h(x)$ is constant and nonzero on U . If there exists such nonempty U , then, by the above discussion, the term $H(x, \cdot)$ is integrable over A_x for each $x \in U$ and hence for each $x \in A'$. If $h(x)$ is almost everywhere zero, then we can, by partitioning A' further using Lemma

2.5, reduce to the case that A' has zero measure or $h(x)$ is identically zero on A' , in which case we can skip the term H . This proves the claim.

Suppose that the statements of the claim are fulfilled for our partition of K^{r+1} into cells. Let \mathcal{P} be the set of cells A such that $\pi_r(A)$ has measure zero. Put $B := \cup_{A \in \mathcal{P}} \pi_r(A)$ and $C := \cup_{A \in \mathcal{P}} A$. Let G' be the constructible function $G(1 - \chi_C)$ where χ_C is the characteristic function of C . Then, B has measure zero in K^r and G' satisfies

$$\int_K G(x, y) |dy| = \int_K G'(x, y) |dy|$$

for all $x \in K^r \setminus B$ and $G'(x, \cdot)$ is integrable for all $x \in K^r$. Putting $F := I_r(G')$, an application of Theorem 2.4 ends the proof. \square

3. EXPONENTIAL SUMS AS FOURIER TRANSFORMS

We fix a nontrivial additive character ψ on K . For $\phi \in \mathcal{C}(K^n)$ an integrable function, for $f : K^n \rightarrow K^r$ a subanalytic function, and for $y \in K^r$, we consider the exponential integral

$$E_{\phi, f}(y) = \int_{K^n} \phi(x) \psi(y \cdot f(x)) |dx|.$$

We call $z \in K^r$ a *regular value* of f if $f^{-1}(z)$ is nonempty, if f is C^1 on a neighborhood of $f^{-1}(z)$, and if the rank of the Jacobian matrix of f is maximal at each point $x \in f^{-1}(z)$. We denote the set of regular values of f by Reg_f and the support of ϕ by $\text{Supp } \phi$.

Theorem 3.1. *Let $f : K^n \rightarrow K^r$ be a subanalytic function and let $\phi \in \mathcal{C}(K^n)$ be an integrable function satisfying $\text{Supp } \phi \subset f^{-1}(\text{Reg}_f) \cup A$ with A a set of measure zero. Then there exists an integrable function $F_{\phi, f}$ in $\mathcal{C}(K^r)$ such that for any bounded continuous function $G : K^r \rightarrow \mathbb{C}$ one has*

$$\int_{K^r} F_{\phi, f}(z) G(z) |dz| = \int_{K^n} \phi(x) G(f(x)) |dx|,$$

and hence, the following Fourier transformation formula holds:

$$E_{\phi, f}(y) = \int_{z \in K^r} F_{\phi, f}(z) \psi(z \cdot y) |dz|.$$

Theorem 3.1 is a generalisation of Corollary 1.8.2 in [7] by Denef which treats the case that the f_i are polynomials and ϕ is a Schwartz-Bruhat function. Igusa has given an analogon of Theorem 3.1 in the case of $r = 1$ polynomial (cf. the asymptotic expansions of [10]), and Lichtin [12] in the case of $r = 2$ polynomials, both in the case that ϕ is a Schwartz-Bruhat function. Igusa and Lichtin also relate the asymptotic expansions to the numerical data of an embedded resolution of f , the counterpart (however not easily computable) of which would be here to apply cell decomposition to get explicit asymptotic expansions for given f and ϕ .

Note that $F_{\phi, f}$ is determined up to a set of measure zero by the universal property stated in the Theorem. The function $F_{\phi_{\text{triv}}, f}$, with ϕ_{triv} the characteristic function of K^n and f a dominant polynomial mapping, is called the local singular series of f and plays an important role in number theory, for example in the circle method.

Proof of Theorem 3.1. Clearly $f^{-1}(\text{Reg}_f)$ is subanalytic. Without loss of generality we may assume that for all $x \in f^{-1}(\text{Reg}_f)$ one has

$$J(x) := \det \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{i,j=1,\dots,r} \neq 0.$$

By the inverse function theorem, Proposition 1.5, Theorem (3.2) of [8] on the existence of bounds, and the subanalytic selection Theorem (3.6) of [8], we may also suppose that

$$T : f^{-1}(\text{Reg}_f) \rightarrow K^n : x \mapsto y = (f(x), x_{r+1}, \dots, x_n)$$

is injective and a C^1 bijection onto its image with C^1 inverse. Applying the change of variables formula, we obtain

$$\begin{aligned} \int_{K^n} \phi(x) G(f(x)) |dx| &= \int_{f^{-1}(\text{Reg}_f)} \phi(x) G(f(x)) |dx| \\ &= \int_{T(f^{-1}(\text{Reg}_f))} \phi \circ T^{-1}(y) G(y_1, \dots, y_r) |J \circ T^{-1}(y)|^{-1} |dy|. \end{aligned}$$

By Fubini's theorem and Proposition 2.7, there exists a function $F_{\phi,f}$ in $\mathcal{C}(K^r)$ with the property that

$$F_{\phi,f}(y_1, \dots, y_r) = \int_{K^{n-r}} \phi \circ T^{-1}(y) |J \circ T^{-1}(y)|^{-1} |dy_{r+1} \wedge \dots \wedge dy_n|,$$

for almost all $(y_1, \dots, y_r) \in K^r$, where we have extended the integrand by zero to K^{n-r} . This function clearly satisfies the requirements of the theorem. \square

4. ESTIMATES FOR FOURIER TRANSFORMS

For an integrable function G in $\mathcal{C}(K^r)$ we write G^* for its Fourier transform

$$G^* : K^r \rightarrow \mathbb{C} : y \mapsto \int_{K^r} G(x) \psi(x \cdot y) |dx|.$$

The following is a generalisation of Theorem 1.2.

Theorem 4.1. *For each integrable $G \in \mathcal{C}(K^r)$ there exists a real number $\alpha < 0$ such that $G^*(y) \ll \min\{|y|^\alpha, 1\}$.*

Proof. For simplicity we suppose that $\psi(R) = 1$ and $\psi(x) \neq 1$ for $x \notin R$ (any other additive character is of the form $x \mapsto \psi(ax)$ with $a \in K$). It is clear that $G^*(y) \ll 1$ since

$$|G^*(y)| \leq \int_{K^r} |G(x)| |dx| < \infty.$$

Hence, it is enough to prove for $i = 1, \dots, r$ that

$$G^*(y) \ll |y_i|^{\alpha_i}$$

for some $\alpha_i < 0$. We prove that $G^*(y) \ll |y_r|^{\alpha_r}$ for some $\alpha_r < 0$. Write $x = (\hat{x}, x_r)$ with $\hat{x} = (x_1, \dots, x_{r-1})$. By Lemma 2.5, we can partition K^r into cells A with center c and coset λP_m such that $G|_A$ is a finite sum of functions of the form

$$(7) \quad H(x) = |(x_r - c(\hat{x}))^a \lambda^{-a}|^{\frac{1}{m}} v(x_r - c(\hat{x}))^s h(\hat{x}),$$

where $h : K^{r-1} \rightarrow \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers.

Claim 2. *Possibly after refining the partition, we can assure that for each A either the projection $A' := \pi_{r-1}(A) \subset K^r$ has zero measure, or we can write $G|_A$ as a sum of terms H of the form (7) such that H is integrable over A and $H(\hat{x}, \cdot)$ is integrable over $A_{\hat{x}} := \{x_r \mid (\hat{x}, x_r) \in A\}$ for all $\hat{x} \in A'$. Moreover, doing so we can assure that each such term H does not change its sign on A .*

As this claim and its proof are similar to Claim 1 we will give only an indication of its proof.

Partitioning further, we may suppose that $v(x_r - c(\hat{x}))$ does not change its sign on A , and that it either takes only one value on A or infinitely many values. If $v(x_r - c(\hat{x}))$ only takes one value on A , we may suppose that the exponents a and s as in (7) are zero. Now apply Lemma 2.5 to each h and to the norms of all the subanalytic functions appearing in the description of the cells A in a similar way (in particular, make similar assumptions as above). Do this inductively for each variable. This way, the claim is reduced to a summation problem over (Presburger set of) integers, which is easily solved (cf. the proof of Claim 1). This proves the claim.

Fix a cell A and a term H as in the claim. The cell A has by definition the following form

$$A = \{x \mid \hat{x} \in A', v(\alpha(\hat{x})) \square_1 v(x_r - c(\hat{x})) \square_2 v(\beta(\hat{x})), \\ x_r - c(\hat{x}) \in \lambda P_m\},$$

where $A' = \pi_{r-1}(A)$ is a cell, \square_i is $<$ or no condition, and $\alpha, \beta : K^{r-1} \rightarrow K^\times$ and $c : K^{r-1} \rightarrow K$ are subanalytic functions. We focus on a cell A with nonempty interior, in particular, $\lambda \neq 0$ and A' has nonempty interior. For $\hat{x} \in A'$ and $y \in K^r$, we denote by $I(\hat{x}, y)$ the value

$$I(\hat{x}, y) = \int_{x_r \in A_{\hat{x}}} H(x) \psi(x \cdot y) |dx_r|.$$

Let $\chi_{\lambda P_m} : K \rightarrow \mathbb{Q}$ be the characteristic function of λP_m and write $\hat{y} = (y_1, \dots, y_{r-1})$. We easily find that $I(\hat{x}, y)$ equals

$$(8) \quad \psi(\hat{x} \cdot \hat{y} + c y_r) h(\hat{x}) |\lambda|^{-a/m} \sum_{(9)} q^{-ja/m} j^s \int_{v(x_r - c) = j} \chi_{\lambda P_m}(x_r - c) \psi((x_r - c) y_r) |dx_r|,$$

where $c = c(\hat{x})$ and the summation is over

$$(9) \quad \{j \mid v(\alpha(\hat{x})) \square_1 j \square_2 v(\beta(\hat{x}))\}.$$

By Hensel's Lemma, there exists an integer e such that all units u with $u \equiv 1 \pmod{\pi_0^e}$ are m -th powers (here, π_0 is such that $v(\pi_0) = 1$). Hence,

$$\int_{v(u)=j} \chi_{\lambda P_m}(u) \psi(u y_r) |du|$$

is zero whenever $j + v(y_r) + e < 0$ (since in this case one essentially sums a nontrivial character over a finite group). By consequence, the only terms contributing to the sum (8) are those for which $-v(y_r) - e \leq j$.

We thus have

$$(10) \quad \left| \int_{x \in A} H(x) \psi(x \cdot y) |dx| \right| = \left| \int_{\hat{x} \in A'} I(\hat{x}, y) |d\hat{x}| \right|$$

$$(11) \quad \leq \int_{B_{y_r}} |H(x)| |dx|$$

with $B_{y_r} = \{x \in K^r \mid x \in A, -v(y_r) - e \leq v(x_r - c(\hat{x}))\}$.

The integrability of H over A , the fact that H does not change its sign on A , and Theorem 2.4 imply that the integral (11), considered as a function in the variable y_r , is in $\mathcal{C}(K)$.

Next we prove that (11) goes to zero when $|y_r|$ goes to infinity. First suppose that A is contained in a compact set. Since $B_{y_r} \subset A$, the measure of B_{y_r} , and hence also (11), goes to zero when $|y_r|$ tends to infinity. In the case that A is not contained in a compact set, let A_b be the intersection of A with $(\pi_0^b R)^r$, for $b < 0$. Clearly each A_b is contained in a compact set. Also, for each $\varepsilon > 0$, there exists a b_0 such that for each $b < b_0$ and for each y_r one has $\int_{B_{y_r} \setminus A_b} |H(x)| |dx| < \varepsilon$, by the integrability of H over A . By the previous discussion, $\int_{B_{y_r} \cap A_b} |H(x)| |dx|$, and hence also (11), goes to zero when $|y_r|$ goes to ∞ .

An application of Corollary 2.6 now finishes the proof. \square

Remark 4.2. The fact that $|G^*|$ in Theorem 4.1 goes to zero when $|y|$ goes to infinity also follows directly from the Lemma of Riemann-Lebesgue in general Fourier analysis, cf. the section on Fourier transforms in [14]. However, to know this is not enough to apply Corollary 2.6 as is done to finish the proof of Theorem 4.1 since in general $|G^*|$ is not subanalytic constructible.

5. DECAY RATES OF EXPONENTIAL SUMS

We use the notation of section 3 for $E_{\phi, f}$. Combining Theorem 4.1 with the Fourier transformation formula of Theorem 3.1, we obtain the following generalization of Theorem 1.3:

Theorem 5.1. *If $f : K^n \rightarrow K^r$ is a subanalytic map and $\phi \in \mathcal{C}(K^n)$ is integrable and satisfies $\text{Supp } \phi \subset f^{-1}(\text{Reg}_f) \cup A$ with A a set of measure zero, then there exists a real number $\alpha < 0$ such that*

$$E_{\phi, f}(y) \ll \min\{|y|^\alpha, 1\}.$$

Combining this Theorem with the fact that the set of singular points of a dominant polynomial mapping $K^n \rightarrow K^r$ (or a dominant restricted analytic mapping $R^n \rightarrow K^r$) has measure zero, we find:

Corollary 5.2. *If $f : K^n \rightarrow K^r$ is a dominant polynomial mapping and if $\phi \in \mathcal{C}(K^n)$ is integrable, then there exists $\alpha < 0$ such that*

$$E_{\phi, f}(y) \ll \min\{|y|^\alpha, 1\}.$$

The same conclusion holds for $E_{\phi, f}$ with $f : R^n \rightarrow K^r$ a restricted analytic map, extended by zero to a map $K^n \rightarrow K^r$, such that $f(R^n)$ has nonempty interior in K^r .

5.1. We end this section with an open question. Let $f = (f_1, \dots, f_r) : K^n \rightarrow K^r$ be an analytic map given by r power series $f_1, \dots, f_r \in K[[x]]$ which converge on the whole of K^n . Suppose that $\phi \in \mathcal{C}(K^n)$ is integrable and that $f(K^n)$ contains a nonempty open. The question is whether there exists an $\alpha < 0$ such that

$$\int_{K^n} \phi(x) \psi(y \cdot f(x)) |dx| \ll \min\{|y|^\alpha, 1\}.$$

6. POLYNOMIAL MAPPINGS

In this section we use elementary methods to deduce explicit upper bounds for polynomial exponential sums. Theorem 6.1 below is of a different nature than our main Theorem 5.1 (and its proof is much more easy), in the sense that it uses the degree of the polynomial mapping as exponent in the upper bound. Such bound based on the degree would give a trivial bound when naively adapted to the analytic case. Similar problems occur when the explicit bounds of Loxton [13] are naively adapted to the analytic case. Since we use a result of [1] formulated there for polynomials over \mathbb{Z} , we will work over \mathbb{Q}_p .

For g a polynomial in $\mathbb{Q}_p[x]$ with $x = (x_1, \dots, x_n)$ let $d_j(g)$ be the degree of g with respect to the variable x_j for $j = 1, \dots, n$, and let $e(g)$ be the minimum of the p -adic orders of the coefficients of $g(x) - g(0)$. For $f = (f_1, \dots, f_r)$ a tuple of polynomials in $\mathbb{Q}_p[x]$ let $d(f)$ be $\max_{i,j} (d_j(f_i))$.

A function $\phi : \mathbb{Q}_p^n \rightarrow \mathbb{Q}$ is a Schwartz-Bruhat function if it is locally constant and has compact support. In this section we consider

$$E_{\phi,f}(y) = \int_{\mathbb{Q}_p^n} \phi(x) \psi(y \cdot f(x)) |dx|,$$

with $f = (f_1, \dots, f_r)$ a tuple of polynomials in $\mathbb{Q}_p[x]$, $\phi : \mathbb{Q}_p^n \rightarrow \mathbb{Q}$ a Schwartz-Bruhat function, ψ a nontrivial additive character on \mathbb{Q}_p , and $y \in \mathbb{Q}_p^r$.

By elementary methods we easily deduce the following from work of Chubarikov [1].

Theorem 6.1. *Suppose that f_1, \dots, f_r are polynomials in $x = (x_1, \dots, x_n)$ over \mathbb{Q}_p which satisfy that $\sum_i a_i f_i + a_0 = 0$ implies $a_i = 0$ for $a_i \in \mathbb{Q}_p$ and $i = 0, \dots, n$. Let $\phi : \mathbb{Q}_p^n \rightarrow \mathbb{Q}$ be a Schwartz-Bruhat function. Then, for any $\varepsilon > 0$, one has*

$$E_{\phi,f}(y) \ll \min\{|y|^{\varepsilon-1/d(f)}, 1\}.$$

Moreover, for y with $v(y) < 0$, one has

$$E_{\phi,f}(y) \ll (-v(y))^{n-1} |y|^{-1/d(f)}.$$

Proof. For simplicity we may assume that $\psi(\mathbb{Z}_p) = 1$ and $\psi(x) \neq 1$ for $x \notin \mathbb{Z}_p$ and that at least one coefficient of $f_1(x) - f_1(0)$ has p -adic order 0. Since ϕ is a finite linear combination of characteristic functions of compact balls, we may moreover assume that ϕ is ϕ_{triv} , that is, the characteristic function of \mathbb{Z}_p^n . Chubarikov [1], Lemma 3, proves that for any polynomial $g \in \mathbb{Z}[x]$ with $e(g) = 0$, $d(g) \leq d$ for some $d \in \mathbb{N}$, and each $z \in \mathbb{Q}_p$ with $v(z) < 0$ one has

$$|E_{\phi_{\text{triv}},g}(z)| < c(d,n) (-v(z))^{n-1} |z|^{-1/d}$$

with $c(d,n)$ a constant only depending on d and n .

Rewrite $E_{\phi,f}(y)$ as

$$E'(z, u_1, \dots, u_r) = \int_{\mathbb{Z}_p^n} \psi(z(u \cdot f(x))) |dx|,$$

with $z \in \mathbb{Q}_p$, $u \in \mathbb{Z}_p^r$ with $|u| = 1$, and $y = (zu_1, \dots, zu_r)$. For any such u , the number $d(u \cdot f)$ cannot exceed $d(f)$. By the compactness and completeness of $\{u \in \mathbb{Z}_p^r \mid |u| = 1\}$, also the number $e(u \cdot f)$ is bounded uniformly in u , say, by N , since otherwise $\sum_i a_i f_i + a_0 = 0$ for some nontrivial $a_i \in \mathbb{Q}_p$.

One easily deduces from the mentioned result of [1] that for $v(z) < -N$

$$|E'(z, u_1, \dots, u_r)| < c(d(f), n) p^{N/d(f)} (-v(z))^{n-1} |z|^{-1/d(f)},$$

with $c(d(f), n)$ as above. The Theorem follows. \square

Remark 6.2. Note that from the proof of Theorem 6.1 and Lemma 3 of [1], one can construct a (non optimal) constant c , depending only on ψ, ϕ , and f , such that for each y with $v(y) < 0$

$$|E_{\phi,f}(y)| < c(-v(y))^{n-1} |y|^{-1/d(f)}.$$

We leave the determination of the optimal c for the future.

Acknowledgment. I would like to thank J. Denef, F. Loeser, and B. Lichtin for interesting conversations on this and related subjects.

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